PART III: Appendix

Appendix A

DERIVATION OF DIVIDEND DISCOUNT MODEL

I. Summation of Infinite Geometric Series

Summation of geometric series can be defined as:

$$S = A + AR + AR^2 + \cdots + AR^{n-1} \tag{A1}$$

Multiplying both sides of equation (A1) by R, we obtain

$$RS = AR + AR^2 + \cdots + AR^{n-1} + AR^n \tag{A2}$$

Subtracting equation (A1) by equation (A2), we obtain

$$S - RS = A - AR^n$$

It can be shown

$$S = \frac{A(1 - R^n)}{1 - R} \tag{A3}$$

If R is smaller than 1, and n approaches to ∞, then R^n approaches to 0 i.e.,

$$S_\infty = A + AR + AR^2 + \cdots + AR^{n-1} + \cdots + AR^\infty, \tag{A4}$$

then,

$$S_\infty = \frac{A}{1 - R} \tag{A5}$$

II. Dividend Discount Model

Dividend Discount Model can be defined as:

$$P_0 = \frac{D_1}{1+k} + \frac{D_2}{(1+k)^2} + \frac{D_3}{(1+k)^3} + \cdots \tag{A6}$$

Where P_0 = present value of stock price per share
D_t = dividend per share in period t ($t = 1, 2, \ldots$,n)

If dividends grow at a constant rate, say g, then, $D_2 = D_1(1 + \text{g})$, $D_3 = D_2(1+g) = D_1(1+g)^2$, and so on.

Then, equation (A6) can be rewritten as:

$$P_0 = \frac{D_1}{1+k} + \frac{D_1(1+g)}{(1+k)^2} + \frac{D_1(1+g)^2}{(1+k)^3} + \cdots \quad \text{or,}$$

$$P_0 = \frac{D_1}{1+k} + \frac{D_1}{(1+k)} \times \frac{(1+g)}{(1+k)} + \frac{D_1}{(1+k)} \times \frac{(1+g)^2}{(1+k)^2} + \cdots \tag{A7}$$

Comparing equation (A7) with equation (A4), i.e., $P_0 = S_\infty$, $\frac{D_1}{1+k} = A$, and $\frac{1+g}{1+k} = R$ as in the equation (A4).

Therefore, if $\frac{1+g}{1+k} < 1$ or if $k > g$, we can use equation (A5) to find out P_0

i.e.,

$$\begin{aligned} P_0 &= \frac{D_1/(1+k)}{1 - [(1+g)/(1+k)]} \\ &= \frac{D_1/(1+k)}{[1 + k - (1+g)]/(1+k)} \\ &= \frac{D_1/(1+k)}{(k-g)/(1+k)} \\ &= \frac{D_1}{k-g} = \frac{D_0(1+g)}{k-g} \end{aligned}$$

Appendix B

DERIVATION OF DOL, DFL AND DCL

I. DOL

Let P = price per unit

V = variable cost per unit

F = total fixed cost

Q = quantity of goods sold

The definition of DOL can be defined as:

DOL(Degree of operating leverage)

$$= \frac{\textit{Percentage Change in Profits}}{\textit{Percentage Change in Sales}}$$

$$= \frac{\Delta\ \text{EBIT}/\text{EBIT}}{\Delta Sales/Sales}$$

$$= \frac{\{[Q(P-V)-F]-[Q'(P-V)-F]\}/[Q(P-V)-F]}{(P\times Q-P\times Q')/(P\times Q)}$$

$$= \frac{[Q(P-V)-Q'(P-V)]/[Q(P-V)-F]}{P(Q-Q')/P\times Q}$$

$$= \frac{(Q-Q')(P-V)/[Q(P-V)-F]}{P(Q-Q')/P\times Q}$$

$$= \frac{\cancel{(Q-Q')}\,(P-V)}{Q\,(P-V)-F}\times\frac{\cancel{P}\times Q}{\cancel{P}\cancel{(Q-Q')}}$$

$$= \boxed{\frac{Q(P-V)}{Q(P-V)-F}}$$

$$= \frac{Q(P-V)-F+F}{Q(P-V)-F} = \frac{Q(P-V)-F}{Q(P-V)-F} + \frac{F}{Q(P-V)-F}$$

$$= 1 + \frac{F}{Q(P-V)-F}$$

$$\boxed{= 1 + \frac{\text{Fixed Costs}}{\textit{Profits}}}$$

II. DFL

Let i = interest rate on outstanding debt; D = outstanding debt } iD = interest payment on dept

N = the total number of shares outstanding

τ = corporate tax rate

EAIT = $[Q(P-V)-F-i\text{D}](1-\tau)$

The definition of DFL can be defined as:

DFL (Degree of financial leverage)

$$= \frac{\Delta\ \text{EPS}/\text{EPS}}{\Delta\ EBIT/EBIT} = \frac{(\Delta\ EAIT/N)/(\text{EAIT}/N)}{\Delta\ \text{EBIT}/\text{EBIT}}$$

$$= \frac{\Delta\ \text{EAIT}/\text{EAIT}}{\Delta\ EBIT/EBIT}$$

$$= \frac{\dfrac{[Q(P-V)-F-i\text{D}](1-\tau)-[Q'(P-V)-F-i\text{D}]}{(1-\tau)[Q(P-V)-F-i\text{D}](1-\tau)}}{\dfrac{[Q(P-V)-F]-[Q'(P-V)-F]}{[Q(P-V)-F]}}$$

$$= \frac{\dfrac{[Q(P-V)](1-\tau)-[Q'(P-V)](1-\tau)}{[Q(P-V)-F-i\text{D}]-(1-\tau)}}{\dfrac{[Q(P-V)-Q'(P-V)]}{[Q(P-V)-F]}}$$

$$= \frac{[\cancel{(Q-Q')\,(P-V)}]\ \cancel{(1-\tau)}}{[Q\,(P-V)-F-i\text{D}]\cancel{(1-\tau)}}\times\frac{Q\,(P-V)-F}{\cancel{(Q-Q')\,(P-V)}}$$

$$= \boxed{\frac{Q(P-V)-F}{Q(P-V)-F-i\text{D}}}\quad\left(=\frac{\text{EBIT}}{EBIT-i\text{D}}\right)$$

III. DCL (degree of combined leverage)

= DOL × DFL

$$= \frac{Q\,(\text{P}-\text{V})}{\cancel{Q\,(\text{P}-\text{V})-\text{F}}}\times\frac{\cancel{Q\,(\text{P}-\text{V})-\text{F}}}{Q\,(\text{P}-\text{V})-\text{F}-i\text{D}} = \boxed{\frac{Q(P-V)}{Q(P-V)-F-i\text{D}}}$$

Appendix C

DERIVATION OF CROSSOVER RATE

Suppose there are 2 projects under consideration. Cash flows of project A, B and B – A are as follows:

Period	0	1	2	3
Project A	−10 500	10 000	1000	1000
Project B	−10 500	1000	1000	12 000
Cash flows of B – A	0	−9000	0	11 000

Based upon the information the table above we can calculate the NPV of Project A and Project B under different discount rates. The results are presented in table C1.

Table C1. NPV of Project A and B under Different Discount Rates

Discount rate	NPV (Project A)	NPV (Project B
0%	1500.00	3500.00
5%	794.68	1725.46
10%	168.67	251.31
15%	−390.69	−984.10
20%	−893.52	−2027.78

NPV(B) is higher with low discount rates and NPV(A) is higher with high discount rates. This is because the cash flows of project A occur early and those of project B occur later. If we assume a high discount rate, we would favor project A; if a low discount rate is expected, project B will be chosen. In order to make the right choice, we can calculate the crossover rate. If the discount rate is higher than the crossover rate, we should choose project A; if otherwise, we should go for project B. **The crossover rate, R_c, is the rate such that NPV(A) equals to NPV(B).**

Suppose the crossover rate is R_c, then

$$\text{NPV}(A) = -10{,}500 + 10{,}000/(1+R_c) + 1{,}000/(1+R_c)^2 + 1{,}000/(1+R_c)^3 \tag{C1}$$

$$\text{NPV}(B) = -10{,}500 + 1{,}000/(1+R_c) + 1{,}000/(1+R_c)^2 + 12{,}000/(1+R_c)^3 \tag{C2}$$

$$\text{NPV}(A) = \text{NPV}(B)$$

Therefore,

$$\boxed{\begin{aligned} &-10{,}500 + \frac{10{,}000}{1+R_c} + \frac{1{,}000}{(1+R_c)^2} + \frac{1{,}000}{(1+R_c)^3} \\ &= -10{,}500 + \frac{1{,}000}{1+R_c} + \frac{1{,}000}{(1+R_c)^2} + \frac{12{,}000}{(1+R_c)^3} \end{aligned}}$$

Rearranging the above equation (moving all terms on the LHS to the RHS), we obtain (C3)

$$\begin{aligned} 0 = {} & [-10{,}500 - (-10{,}500)] + \left[\frac{1{,}000}{1+R_c} - \frac{10{,}000}{1+R_c}\right] \\ & + \left[\frac{1{,}000}{(1+R_c)^2} - \frac{1{,}000}{(1+R_c)^2}\right] + \left[\frac{12{,}000}{(1+R_c)^3} - \frac{1{,}000}{(1+R_c)^3}\right] \end{aligned} \tag{C3}$$

Solving equation (C3) by trial and error method for R_c, R_c equals 10.55%.

Using the procedure of calculating internal rate of return (IRR) as discussed in equations (C1), (C2), and (C3), we calculate the IRR for both Project A and Project B. The IRR for Project A and B are 11.45% and 10.95% respectively. From this information, we have concluded that Project A will perform better than Project B without consideration for change of discount rate. Therefore, the IRR decision rule cannot be used for capital budgeting decisions when there exists an increasing or decreasing net cash inflow. This is so called "The Timing Problem" for using the IRR method for capital budgeting decisions.

Appendix D

CAPITAL BUDGETING DECISIONS WITH DIFFERENT LIVES

I. Mutually Exclusive Investment Projects with Different Lives

The traditional NPV technique may not be the appropriate criterion to select a project from mutually exclusive investment projects, if these projects have different lives. The underlying reason is that, compared with a long-life project, a short-life project can be replicated more quickly in the long run. In order to compare projects with different lives, we compute the NPV of an infinite replication of the investment project. For example, let Projects A and B be two mutually exclusive investment projects with the following cash flows.

Year	Project A	Project B
0	100	100
1	70	50
2	70	50
3		50

By assuming a discount rate of 12 percent, the traditional NPV of Project A is 18.30 and the NPV of Project B is 20.09. This shows that Project B is a better choice than Project A. However, the NPV with infinite replications for Project A and B should be adjusted into a comparable basis.

In order to compare Projects A and B, we compute the NPV of an infinite stream of constant scale replications. Let NPV (N, ∞) be the NPV of an N-year project with NPV (N), replicated forever. This is exactly the same as an annuity paid at the beginning of the first period and at the end of every N years from that time on. The NPV of the annuity is:

$$NPV(N, \infty) = NPV(N) + \frac{NPV(N)}{(1+K)^N} + \frac{NPV(N)}{(1+K)^{2N}} + \cdots$$

In order to obtain a closed-form formula, let $(1/[(1+K)^N]) = H$. Then we have:

$$NPV(N, t) = NPV(N)(1 + H + H^2 + \cdots + H^t) \quad \text{(D1)}$$

Multiplying both sides by H, this becomes

$$H[NPV(N, t)] = NPV(N)(H + H^2 + \cdots + H^t + H^{t+1}) \quad \text{(D2)}$$

Subtracting equation. (D2) from equation. (D1) gives:

$$NPV(N, t) - (H)NPV(N, t) = NPV(N)(1 - H^{t+1})$$

$$NPV(N, t) = \frac{NPV(N)(1 - H^{t+1})}{1 - H}$$

Taking the limit as the number of replications, t, approaches infinity gives:

$$\lim_{t\to\infty} NPV(N, t) = NPV(N, \infty) = NPV\left[\frac{1}{1 - [1/(1+K)^N]}\right] = NPV(N)\left[\frac{(1+K)^N}{(1+K)^N - 1}\right] \quad \text{(D3)}$$

Equation (D3) is the NPV of an N-year project replicated at constant scale an infinite number of times. We can use it to compare projects with different lives because when their cash-flow streams are replicated forever, it is as if they had the same (infinite) life.

Based upon equation (D3), we can calculate the NPV of Projects A and B as follows:

For Project A

$$NPV(2, \infty) = NPV(2)\left[\frac{(1+0.12)^2}{(1+0.12)^2-1}\right] = (18.30)\left[\frac{1.2544}{0.2544}\right] = 90.23$$

For Project B

$$NPV(3, \infty) = NPV(3)\left[\frac{(1+0.12)^3}{(1+0.12)^3-1}\right] = 20.09\left[\frac{1.4049}{0.4049}\right] = 69.71$$

Consequently, we would choose to accept Project A over Project B, because, when the cash flows are adjusted for different lives, A provides the greater cash flow.

Alternatively, equation (D3) can be rewritten as an equivalent annual NPV version as:

$$K \times NPV(N, \infty) = \frac{NPV(N)}{Annuity\ factor} \tag{D4}$$

where the annuity factor is

$$\frac{1 - 1/(1+K)^N}{K}$$

The decision rule from equation (D4) is equivalent to the decision rule of equation (D3).

The different project lives can affect the beta coefficient estimate, as shown by Meyers and Turnbull (1977). For empirical guidance for evaluating capital-investment alternatives with unequal lives, the readers are advised to refer Emery (1982).

II. Equivalent Annual Cost

Equation (D4) can be written as:

$$NPV(N) = K \times NPV(N, \infty) \times \text{Annuity Factor} \tag{D5}$$

Corporate Finance by Ross, Westerfield, and Jaffe (2005, 7th edn, p. 193) has discussed about Equivalent Annual Cost. The Equivalent Annual Cost (C) can be calculated as follows:

$$NPV(N) = C \times \text{Annuity Factor} \tag{D6}$$

From equation (D5) and (D6), we obtain

$$C = K \times NPV(N, \infty) \tag{D7}$$

Assume company A buys a machine that costs \$1000 and the maintenance expense of \$250 is to be paid at the end of each of the four years. To evaluate this investment, we can calculate the present value of the machine. Assuming the discount rate as 10 percent, we have

$$NPV(A) = 1000 + \frac{250}{1.1} + \frac{250}{(1.1)^2} + \frac{250}{(1.1)^3} + \frac{250}{(1.1)^4} = 1792.47 \tag{D8}$$

Equation (D8) shows that payments of (1000, 250, 250, 250, 250) are equivalent to a payment of 1792.47 at time 0. Using equation (D6), we can equate the payment at time 0 of 1792.47 with a four year annuity.

$$1792.47 = C \times A^4_{0.1} = C \times 3.1699$$
$$C = 565.47$$

In this example, following equation (D3), we can find

$$NPV(N, \infty) = 1749.47 \times (1+0.1)^4/[(1+0.1)^4 - 1] = 5654.71$$

Then following the equation (D7), we obtain

$$C = K \times NPV(N, \infty) = 0.1 \times 5654.71 = 565.47$$

Therefore, the equivalent annual cost C is identical to the equivalent annual NPV as defined in equation (D4).

Appendix E

DERIVATION OF MINIMUM-VARIANCE PORTFOLIO

If there is a two security portfolio, its variance can be defined as:

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_Dw_E\text{Cov}(r_D,r_E) \qquad \text{(E1)}$$

where r_D and r_E are the rate of return for security D and security E respectively; w_D and w_E are weight associated with security D and E respectively; σ_D^2 and σ_E^2 are variance of security D and E respectively; and Cov(r_D, r_E) is the covariance between r_D and r_E.

The problem is choosing optimal w_D to minimize the portfolio variance, σ_p^2

$$\underset{w_D}{Min}\ \sigma_P^2 \qquad \text{(E2)}$$

We can solve the minimization problem by differentiating the σ_p^2 with respect to w_D and setting the derivative equal to 0 i.e., we want to solve

$$\frac{\partial\sigma_p^2}{\partial w_D} = 0 \qquad \text{(E3)}$$

Since, $w_D + w_E = 1$ or, $w_E = 1 - w_D$

therefore, the variance, σ_p^2, can be rewritten as

$$\begin{aligned}\sigma_p^2 &= w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_Dw_E\ \text{Cov}(r_D,r_E)\\ &= w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\ \text{Cov}(r_D,r_E)\\ &= w_D^2\sigma_D^2 + \sigma_E^2 - 2w_D\sigma_E^2 + w_D^2\sigma_E^2 + 2w_D\ \text{Cov}(r_D,r_E)\\ &\quad - 2w_D^2\text{Cov}(r_D,r_E)\end{aligned}$$

Now, the first order conditions of equation (E3) can be written as

$$2w_D\sigma_D^2 - 2\sigma_E^2 + 2w_D\sigma_E^2 + 2\ \text{Cov}(r_D,r_E) - 4w_D\ \text{Cov}(r_D,r_E) = 0$$

Rearranging the above equation,

$$\begin{aligned}w_D\sigma_D^2 + w_D\sigma_E^2 - 2w_D\ \text{Cov}(r_D,r_E) &= \sigma_E^2 - \text{Cov}(r_D,r_E)\\ \left[\sigma_D^2 + \sigma_E^2 - 2\text{Cov}(r_D,r_E)\right]w_D &= \sigma_E^2 - \text{Cov}(r_D,r_E)\end{aligned}$$

Finally, we have

$$\boxed{w_D = \frac{\sigma_E^2 - \text{Cov}(r_D,r_E)}{\sigma_D^2 + \sigma_E^2 - 2\text{Cov}(r_D,r_E)}}$$

Appendix F

DERIVATION OF AN OPTIMAL WEIGHT PORTFOLIO USING THE SHARPE PERFORMANCE MEASURE

Solution for the weights of the optimal risky portfolio can be found by solving the following maximization problem:

$$\underset{w_D}{Max} S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

where $E(r_p) =$ expected rates of return for portfolio P

$r_f =$ risk free rates of return

$S_p =$ sharpe performance measure, and

σ_p as defined in equation (E1) of Appendix E

We can solve the maximization problem by differentiating the S_p with respect to w_D, and setting the derivative equal to 0 i.e., we want to solve

$$\frac{\partial S_p}{\partial w_D} = 0 \tag{F1}$$

In the case of two securities, we know that

$$E(r_p) = w_D\ E(r_D) + w_E\ E(r_E) \tag{F2}$$

$$\sigma_p = \left[w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D\ w_E\ \text{Cov}(r_D,\ r_E)\right]^{1/2} \tag{F3}$$

$$w_D + w_E = 1 \tag{F4}$$

From above equations (F2), (F3), and (F4), we can rewrite $E(r_p) - r_f$ and σ_p as:

$$\begin{aligned} E(r_p) - r_f &= w_D\ E(r_D) + w_E\ E(r_E) - r_f \\ &= w_D\ E(r_D) + (1 - w_D)\ E(r_E) - r_f \\ &\equiv \boldsymbol{f(w_D)} \end{aligned} \tag{F5}$$

$$\begin{aligned} \sigma_p &= \left[w_D^2\ \sigma_D^2 + w_E^2\ \sigma_E^2 + 2w_D w_E\ \text{Cov}(r_D,\ r_E)\right]^{1/2} \\ &= \left[w_D^2\ \sigma_D^2 + (1 - w_D)^2\ \sigma_E^2 + 2w_D(1 - w_D)\ \text{Cov}(r_D,\ r_E)\right]^{1/2} \\ &\equiv \boldsymbol{g(w_D)} \end{aligned} \tag{F6}$$

Equation (F1) becomes

$$\begin{aligned} \frac{\partial S_p}{\partial w_D} &= \frac{\partial[f(w_D)/g(w_D)]}{\partial w_D} \\ &= \frac{f'(w_D)g(w_D) - f(w_D)g'(w_D)}{[g(w_D)]^2} = 0 \end{aligned} \tag{F7}$$

$$\text{where } f'(w_D) = \frac{\partial f(w_D)}{\partial w_D} = E(r_D) - E(r_E) \tag{F8}$$

$$\begin{aligned} g'(w_D) &= \frac{\partial g(w_D)}{\partial w_D} \\ &= \frac{1}{2} \times \left[w_D^2\sigma_D^2 + (1 - w_D)^2\sigma_E^2 + 2w_D(1 - w_D)\ \text{Cov}(r_D,r_E)\right]^{1/2-1} \\ &\quad \times \left[2w_D\sigma_D^2 + 2w_D\sigma_E^2 - 2\sigma_E^2 + 2\text{Cov}(r_D,r_E) - 4w_D\ \text{Cov}(r_D,r_E)\right] \\ &= \left[w_D\sigma_D^2 + w_D\sigma_E^2 - \sigma_E^2 + \text{Cov}(r_D,r_E) - 2w_D\ \text{Cov}(r_D,r_E)\right] \\ &\quad \times \left[w_D^2\sigma_D^2 + (1 - w_D)^2\sigma_E^2 + 2w_D(1 - w_D)\ \text{Cov}(r_D,r_E)\right]^{-1/2} \end{aligned} \tag{F9}$$

From equation (F7),

$$f'(w_D)g(w_D) - f(w_D)g'(w_D) = 0, \text{ or } f'(w_D)g(w_D) = f(w_D)g'(w_D) \tag{F10}$$

Now, plugging $f(w_D)$, $g(w_D)$, $f'(w_D)$, and $g'(w_D)$ [equations (F5), (F6), (F8), and (F9)] into equation (F10), we have

$$\begin{aligned}&[E(r_D) - E(r_E)] \\ &\times \left[w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\text{Cov}(r_D,r_E)\right]^{1/2} \\ &= \left[w_D E(r_D) + (1-w_D)E(r_E) - r_f\right] \\ &\quad \times \left[w_D\sigma_D^2 + w_D\sigma_E^2 - \sigma_E^2 + \text{Cov}(r_D,r_E) - 2w_D\text{Cov}(r_D,r_E)\right] \\ &\quad \times \left[w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\text{Cov}(r_D,r_E)\right]^{-1/2}\end{aligned} \tag{F11}$$

Multiplying by $\left[w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\text{ Cov}(r_D, r_E)\right]^{1/2}$ on both sides of equation (F11), we have

$$\begin{aligned}&[E(r_D) - E(r_E)] \\ &\times \left[w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\text{Cov}(r_D,r_E)\right] \\ &= \left[w_D E(r_D) + (1-w_D)E(r_E) - r_f\right] \\ &\times \left[w_D\sigma_D^2 + w_D\sigma_E^2 - \sigma_E^2 + \text{Cov}(r_D,r_E) - 2w_D\text{Cov}(r_D,r_E)\right]\end{aligned} \tag{F12}$$

Rearrange all terms on both hand sides of equation (F12), i.e.,

Left hand side of equation (F12)

$$\begin{aligned}&[E(r_D) - E(r_E)] \\ &\quad \times \left[w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\text{Cov}(r_D,r_E)\right] \\ &= [E(r_D) - E(r_E)] \\ &\quad \times \left[w_D^2\sigma_D^2 + \sigma_E^2 - 2w_D\sigma_E^2 + w_D^2\sigma_E^2 + 2w_D\text{Cov}(r_D,r_E) - 2w_D^2\text{Cov}(r_D,r_E)\right] \\ &= [E(r_D) - E(r_E)] \times \left\{w_D^2\left[\sigma_D^2 + \sigma_E^2 - 2\text{Cov}(r_D,r_E)\right] + 2w_D\left[\text{Cov}(r_D,r_E) - \sigma_E^2\right] + \sigma_E^2\right\} \\ &= [E(r_D) - E(r_E)] \times \left\{w_D^2\left[\sigma_D^2 + \sigma_E^2 - 2\text{Cov}(r_D,r_E)\right]\right\} \\ &\quad + [E(r_D) - E(r_E)] \times \left\{2w_D\left[\text{Cov}(r_D,r_E) - \sigma_E^2\right]\right\} + [E(r_D) - E(r_E)] \times \sigma_E^2 = [E(r_D) - E(r_E)] \times \left[\sigma_D^2 + \sigma_E^2 - 2\text{Cov}(r_D,r_E)\right]w_D^2 + 2[E(r_D) - E(r_E)] \\ &\quad \times \left[\text{Cov}(r_D,r_E) - \sigma_E^2\right]w_D + [E(r_D) - E(r_E)] \times \sigma_E^2\end{aligned}$$

Right hand side of equation (F12)

$$\begin{aligned}&[w_D E(r_D) + (1-w_D)E(r_E) - r_f] \times [w_D\sigma_D^2 + w_D\sigma_E^2 - \sigma_E^2 + Cov(r_D,r_E) - 2w_D Cov(r_D,r_E)] \\ &= [w_D E(r_D) + E(r_E) - w_D E(r_E) - r_f] \times [w_D\sigma_D^2 + w_D\sigma_E^2 - 2w_D Cov(r_D,r_E) - \sigma_E^2 + Cov(r_D,r_E)] \\ &= \left\{w_D[E(r_D) - E(r_E)] + [E(r_E) - r_f]\ \right\} \times \left\{w_D[\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)] + Cov(r_D,r_E) - \sigma_E^2\right\} \\ &= w_D[E(r_D) - E(r_E)] \times w_D[\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)] \\ &\quad + w_D[E(r_D) - E(r_E)] \times [Cov(r_D,r_E) - \sigma_E^2] \\ &\quad + [E(r_E) - r_f] \times w_D[\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)] \\ &\quad + [E(r_E) - r_f] \times [Cov(r_D,r_E) - \sigma_E^2] \\ &= [E(r_D) - E(r_E)] \times [\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)]w_D^2 \\ &\quad + [E(r_D) - E(r_E)] \times [Cov(r_D,r_E) - \sigma_E^2]w_D \\ &\quad + [E(r_E) - r_f] \times [\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)]w_D \\ &\quad + [E(r_E) - r_f] \times [Cov(r_D,r_E) - \sigma_E^2]\end{aligned}$$

Subtracting $[E(r_D) - E(r_E)][\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)]w_D^2$ and $[E(r_D) - E(r_E)][Cov\ (r_D,r_E) - \sigma_E^2]w_D$ from both hand sides of equation (F12), we have

$$\begin{aligned}&[E(r_D) - E(r_E)] \times [Cov(r_D,r_E) - \sigma_E^2]w_D + [E(r_D) - E(r_E)] \times \sigma_E^2 \\ &= [E(r_E) - r_f] \times [\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)]w_D \\ &\quad + [E(r_E) - r_f] \times [Cov(r_D,r_E) - \sigma_E^2]\end{aligned} \tag{F13}$$

Moving all the terms with w_D on one side and leaving the rest terms on the other side from equation (F13), we have

$$
\begin{aligned}
&[E(r_D) - E(r_E)] \times \sigma_E^2 - [E(r_E) - r_f] \\
&\quad \times [Cov(r_D,r_E) - \sigma_E^2] \\
&= [E(r_E) - r_f] \times [\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)]w_D \\
&\quad - [E(r_D) - E(r_E)] \times [Cov(r_D,r_E) - \sigma_E^2]w_D
\end{aligned} \tag{F14}
$$

Rearrange equation (F14) in order to solve for w_D, *i.e.*,

$$
\begin{aligned}
&[E(r_D) - E(r_E) + E(r_E) - r_f] \times \sigma_E^2 \\
&\quad - [E(r_E) - r_f]Cov(r_D,r_E) \\
&= \{[E(r_E) - r_f]\sigma_D^2 + [E(r_E) - r_f]\sigma_E^2 \\
&\quad - [E(r_E) - r_f][2Cov(r_D,r_E)] - [E(r_D) \\
&\quad - E(r_E)]Cov(r_D,r_E) + [E(r_D) - E(r_E)]\sigma_E^2\}w_D \\
&= \{[E(r_D) - r_f]\sigma_E^2 + [E(r_E) - r_f]\sigma_D^2 - [E(r_D) \\
&\quad - r_f + E(r_E) - r_f]Cov(r_D,r_E)]\}w_D
\end{aligned}
$$

Finally, we have the optimum weight of security D as

$$
w_D = \frac{[E(r_D) - r_f]\sigma_E^2 - [E(r_E) - r_f]Cov(r_D,r_E)}{\begin{array}{c}[E(r_D) - r_f]\sigma_E^2 + [E(r_E) - r_f]\sigma_D^2 \\ -[E(r_D) - r_f + E(r_E) - r_f]Cov(r_D,r_E)\end{array}}
$$

Appendix G

APPLICATIONS OF THE BINOMIAL DISTRIBUTION TO EVALUATE CALL OPTIONS

In this appendix, we show how the binomial distribution is combined with some basic finance concepts to generate a model for determining the price of stock options.

What is an Option?

In the most basic sense, an **option** is a contract conveying the right to buy or sell a designated security at a stipulated price. The contract normally expires at a predetermined date. The most important aspect of an option contract is that the purchaser is under no obligation to buy; it is, indeed, an "option." This attribute of an option contract distinguishes it from other financial contracts. For instance, whereas the holder of an option may let his or her claim expire unused if he or she so desires, other financial contracts (such as futures and forward contracts) obligate their parties to fulfill certain conditions.

A *call option* gives its owner the right to buy the underlying security, a *put option* the right to sell. The price at which the stock can be bought (for a call option) or sold (for a put option) is known as the exercise price.

The Simple Binomial Option Pricing Model

Before discussing the binomial option model, we must recognize its two major underlying assumptions. First, the binomial approach assumes that trading takes place in discrete time, that is, on a period-by-period basis. Second, it is assumed that the stock price (the price of the underlying asset) can take on only two possible values each period; it can go up or go down.

Say we have a stock whose current price per share S can advance or decline during the next period by a factor of either u (up) or d (down). This price either will increase by the proportion $u-1 \geq 0$ or will decrease by the proportion $1-d$, $0<d<1$. Therefore, the value S in the next period will be either uS or dS. Next, suppose that a call option exists on this stock with a current price per share of C and an exercise price per share of X and that the option has one period left to maturity. This option's value at expiration is determined by the price of its underlying stock and the exercise price X. The value is either

$$C_u = Max(0, uS - X) \tag{G1}$$

or

$$C_d = Max(0, dS - X) \tag{G2}$$

Why is the call worth Max $(0, uS - X)$ if the stock price us uS? The option holder is not obliged to purchase the stock at the exercise price of X, so she or he will exercise the option only when it is beneficial to do so. This means the option can never have a negative value. When is it beneficial for the option holder to exercise the option? When the price per share of the stock is greater than the price per share at which he or she can purchase the stock by using the option, which is the exercise price, X. Thus if the stock price uS exceeds the exercise price X, the investor can exercise the option and buy the stock. Then he or she can immediately sell it for uS, making a profit of $uS–X$ (ignoring commission). Likewise, if the stock price declines to dS, the call is worth Max $(0, dS - X)$.

Also for the moment, we will assume that the risk-free interest rate for both borrowing and lending is equal to r percent over the one time period and that the exercise price of the option is equal to X.

To intuitively grasp the underlying concept of option pricing, we must set up a *risk-free portfolio* – a combination of assets that produces the same return in every state of the world over our chosen investment horizon. The investment horizon is assumed to be one period (the duration of this period can be any length of time, such as an hour, a day, a week, etc.). To do this, we buy h share of the stock and sell the call option at its current price of C. Moreover, we choose the value of h such that our portfolio will yield the same payoff whether the stock goes up or down.

$$h(uS) - C_u = h(dS) - C_d \tag{G3}$$

By solving for h, we can obtain the number of shares of stock we should buy for each call option we sell.

$$h = \frac{C_u - C_d}{(u - d)S} \tag{G4}$$

Here h is called the *hedge ratio*. Because our portfolio yields the same return under either of the two possible states for the stock, it is without risk and therefore should yield the risk-free rate of return, r percent, which is equal to the risk-free borrowing and lending rate, the condition must be true; otherwise, it would be possible to earn a risk-free profit without using any money. Therefore, the ending portfolio value must be equal to $(1 + r)$ times the beginning portfolio value, $hS - C$.

$$(1 + r)(hS - C) = h(uS) - C_u = h(dS) - C_d \tag{G5}$$

Note that S and C represent the beginning values of the stock price and the option price, respectively.

Setting $R = 1 + r$, rearranging to solve for C, and using the value of h from Equation (G4), we get

$$C = \left[\left(\frac{R - d}{u - d}\right)C_u + \left(\frac{u - R}{u - d}\right)C_d\right]/R \tag{G6}$$

where $d < r < u$. To simplify this equation, we set

$$p = \frac{R - d}{u - d} \text{ so } 1 - p = \left\{\frac{u - R}{u - d}\right\} \tag{G7}$$

Thus we get the option's value with one period to expiration

$$C = \frac{pC_u + (1 - p)C_d}{R} \tag{G8}$$

This is the binomial call option valuation formula in its most basic form. In other words, this is the binomial valuation formula with one period to expiration of the option.

To illustrate the model's qualities, let's plug in the following values, while assuming the option has one period to expiration. Let

$X = \$100$
$S = \$100$
$U = (1.10)$, so $uS = \$110$
$D = (0.90)$, so $dS = \$90$
$R = 1 + r = 1 + 0.07 = 1.07$

Table G.1. Possible Option Value at Maturity

Today Stock (S)	Option (C)		Next Period (Maturity)
		$uS = \$110$	$C_u = Max(0, uS - X)$ $= Max(0, 110 - 100)$ $= Max(0, 10)$ $= \$10$
\$100	C		
		$dS = \$90$	$C_d = Max(0, dS - X)$ $= Max(0, 90 - 100)$ $= Max(0, -10)$ $= \$0$

First we need to determine the two possible option values at maturity, as indicated in Table G.1.

Next we calculate the value of p as indicated in Equation (G7).

$$p = \frac{1.07 - 0.90}{1.10 - 0.90} = 0.85 \text{ so } 1 - p = \frac{1.10 - 1.07}{1.10 - 0.90} = 0.15$$

Solving the binomial valuation equation as indicated in Equation (G8), we get

$$C = \frac{0.85(10) + 0.15(0)}{1.07} = \$7.94$$

The correct value for this particular call option today, under the specified conditions, is \$7.94. If the call option does not sell for \$7.94, it will be possible to earn arbitrage profits. That is, it will be possible for the investor to earn a risk-free profit while using none of his or her own money. Clearly, this type of opportunity cannot continue to exist indefinitely.

The Generalized Binomial Option Pricing Model

Suppose we are interested in the case where there is more than one period until the option expires. We can extend the one-period binomial model to consideration of two or more periods.

Because we are assuming that the stock follows a binomial process, from one period to the next it can only go up by a factor of u or go down by a factor of d. After one period the stock's price is either uS or dS. Between the first and second periods, the stock's price can once again go up by u or down by d, so the possible prices for the stock two periods from now are uuS, udS, and ddS. This process is demonstrated in tree diagram (Figure G.1) given in Example G.1 later in this appendix.

Note that the option's price at expiration, two periods from now, is a function of the same relationship that determined its expiration price in the one-period model, more specifically, the call option's maturity value is always

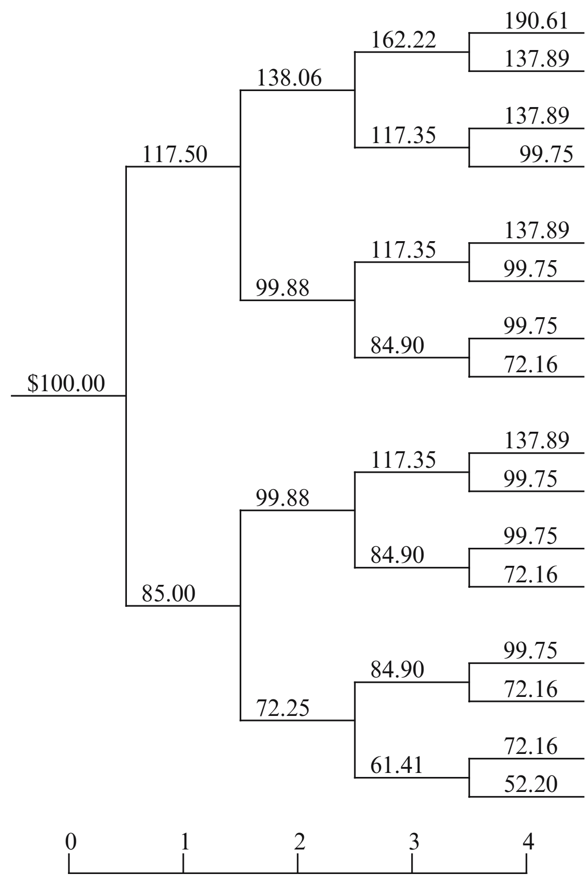

Figure G.1. Price Path of Underlying Stock *Source:* Rendelman, R.J., Jr., and Bartter, B.J. (1979). "Two-State Option Pricing," *Journal of Finance* 34 (December), 1906.

$$C_T = [0,\ S_T - X] \tag{G9}$$

where T designated the maturity date of the option.

To derive the option's price with two periods to go ($T = 2$), it is helpful as an intermediate step to derive the value of C_u and C_d with one period to expiration when the stock price is either uS or dS, respectively.

$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{R} \tag{G10}$$

$$C_d = \frac{pC_{du} + (1-p)C_{dd}}{R} \tag{G11}$$

Equation (G10) tells us that if the value of the option after one period is C_u, the option will be worth either C_{uu} (if the stock price goes up) or C_{ud} (if stock price goes down) after one more period (at its expiration date). Similarly, Equation (G11) shows that the value of the option is C_d after one period, the option will be worth either C_{du} or C_{dd} at the end of the second period. Replacing C_u and C_d in Equation (G8) with their expressions in Equations (G10) and (G11), respectively, we can simplify the resulting equation to yield the two-period equivalent of the one-period binomial pricing formula, which is

$$C = \frac{p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}}{R^2} \tag{G12}$$

In Equation (G12), we used the fact that $C_{ud} = C_{du}$ because the price will be the same in either case.

We know the values of the parameters S and X. If we assume that R, u, and d will remain constant over time, the possible maturity values for the option can be determined exactly. Thus deriving the option's fair value with two periods to maturity is a relatively simple process of working backwards from the possible maturity values.

Using this same procedure of going from a one-period model to a two-period model, we can extend the binomial approach to its more generalized form, with n periods maturity

$$C = \frac{1}{R^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \; Max[0, u^k d^{n-k} S - X] \tag{G13}$$

To actually get this form of the binomial model, we could extend the two-period model to three periods, then from three periods to four periods, and so on. Equation (G13) would be the result of these efforts. To show how Equation (G13) can be used to assess a call option's value, we modify the example as follows: $S = \$100$, $X = \$100$, $R = 1.07$, $n = 3$, $u = 1.1$ and $d = 0.90$.

First we calculate the value of p from Equation (G7) as 0.85, so $1 - p$ is 0.15. Next we calculate the four possible ending values for the call option after three periods in terms of $Max[0, u^k d^{n-k} S - X]$.

$$C_1 = [0, (1.1)^3(0.90)^0(100) - 100] = 33.10$$
$$C_2 = [0, (1.1)^2(0.90)\,(100) - 100] = 8.90$$
$$C_3 = [0, (1.1)\,(0.90)^2(100) - 100] = 0$$
$$C_4 = [0, (1.1)^0(0.90)^3(100) - 100] = 0$$

Now we insert these numbers (C_1, C_2, C_3, and C_4) into the model and sum the terms.

$$\begin{aligned} C &= \frac{1}{(1.07)^3}\left[\frac{3!}{0!3!}(0.85)^0(0.15)^3 \times 0 + \frac{3!}{1!2!}(0.85)^1(0.15)^2 \times 0 + \frac{3!}{2!1!}(0.85)^2(0.15)^1 \times 8.90 + \frac{3!}{3!0!}(0.85)^3(0.15)^0 \times 33.10\right] \\ &= \frac{1}{1.225}\left[0 + 0 + \frac{3\times2\times1}{2\times1\times1}(0.7225)(0.15)(8.90) + \frac{3\times2\times1}{3\times2\times1\times1}\times(0.61413)(1)(33.10)\right] \\ &= \frac{1}{1.225}[(0.32513 \times 8.90) + (0.61413 \times 33.10)] \\ &= \$18.96 \end{aligned}$$

As this example suggests, working out a multiple-period problem by hand with this formula can become laborious as the number of periods increases. Fortunately, programming this model into a computer is not too difficult.

Now let's derive a binomial option pricing model in terms of the cumulative binomial density function. As a first step, we can rewrite Equation (G13) as

$$C = S\left[\sum_{k=m}^{n} \frac{n!}{k!(n-K)!} p^K (1-p)^{n-k} \frac{u^k d^{n-k}}{R^n}\right] - \frac{X}{R^n}\left[\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}\right] \tag{G14}$$

This formula is identical to Equation (G13) except that we have removed the Max operator. In order to remove the Max operator, we need to make $u^k d^{n-k} S - X$ positive, which we can do by changing the counter in the summation from $k = 0$ to $k = m$. What is m? It is the minimum number of upward stock movements necessary for the option to terminate "in the money" (that is, $u^k d^{n-k} S - X > 0$). How can we interpret Equation (G14)? Consider the second term in brackets; it is just a cumulative binomial distribution with parameters of n and p. Likewise, via a small algebraic manipulation we can show that the first term in the brackets is also a cumulative binomial distribution. This can be done by defining $P' \equiv (u/R)p$ and $1 - P' \equiv (d/R)(1-p)$. Thus

$$P^k (1-p)^{n-k} \frac{u^k d^{n-k}}{R^n} = p'^k (1-p')^{n-k}$$

Therefore the first term in brackets is also a cumulative binomial distribution with parameters of n and p'. Using Equation (G10) in the text, we can write the binomial call option model as

$$C = SB_1(n, p', m) - \frac{X}{R^n} B_2(n, p, m) \tag{G15}$$

where

$$B_1(n, p', m) = \sum_{k=m}^{n} C_k^n p'^k (1-p')^{n-k}$$

$$B_2(n, p, m) = \sum_{k=m}^{n} C_k^n p^k (1-p)^{n-k}$$

and m is the minimum amount of time the stock has to go up for the investor to finish *in the money* (that is, for the stock price to become larger than the exercise price).

In this appendix, we showed that by employing the definition of a call option and by making some simplifying assumptions, we could use the binomial distribution to find the value of a call option. In the next chapter, we will show how the binomial distribution is related to the normal distribution and how this relationship can be used to derive one of the most famous valuation equations in finance, the Black-Scholes option pricing model.

Example G.1

A Decision Tree Approach to Analyzing Future Stock Price

By making some simplifying assumptions about how a stock's price can change from one period to the next, it is possible to forecast the future price of the stock by means of a decision tree. To illustrate this point, let's consider the following example.

Suppose the price of Company A's stock is currently \$100. Now let's assume that from one period to the next, the stock can go up by 17.5 percent or go down by 15 percent. In addition, let us assume that there is a 50 percent chance that the stock will go up and a 50 percent chance that the stock will go down. It is also assumed that the price movement of a stock (or of the stock market) today is completely independent of its movement in the past; in other words, the price will rise or fall today by a random amount. A sequence of these random increases and decreases is known as a **random walk**.

Given this information, we can lay out the paths that the stock's price may take. Figure G.1 shows the possible stock prices for company A for four periods.

Note that in period 1 there are two possible outcomes: the stock can go up in value by 17.5 percent to \$117.50 or down by 15 percent to \$85.00. In period 2 there are four possible outcomes. If the stock went up in the first period, it can go up again to \$138.06 or down in the second period to \$99.88. Likewise, if the stock went down in the first period, it can go down again to \$72.25 or up in the second period to \$99.88. Using the same argument, we can trace the path of the stock's price for all four periods.

If we are interested in forecasting the stock's price at the end of period 4, we can find the average price of the stock for the 16 possible outcomes that can occur in period 4.

$$\bar{P} = \frac{\sum_{i=1}^{16} P_i}{16} = \frac{190.61 + 137.89 + \cdots + 52.20}{16} = \$105.09$$

We can also find the standard deviation for the stock's return.

$$\sigma_P = \left[\frac{(190.61 - 105.09)^2 + \cdots + (52.20 - 105.09)^2}{16}\right]^{1/2}$$
$$= \$34.39$$

$\bar{P}$ and σ_P can be used to predict the future price of stock A.